Discrete Mathematics 311 (2011) 289–294

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Matching signatures and Pfaffian graphs

Alberto Alexandre Assis Miranda [a,*], Cláudio Leonardo Lucchesi [a,b]

[a] Institute of Computing, University of Campinas, 13084-971 Campinas, SP, Brazil
[b] FACOM-UFMS, 79070-900 Campo Grande, MS, Brazil

ARTICLE INFO

Article history:
Received 7 June 2010
Received in revised form 17 October 2010
Accepted 21 October 2010
Available online 20 November 2010

Keywords:
Pfaffian graphs
Perfect matchings
Matching covered graphs

ABSTRACT

Perfect matchings of k-Pfaffian graphs may be enumerated in polynomial time on the number of vertices, for fixed k. In general, this enumeration problem is $\#P$-complete. We give a Composition Theorem of $2r$-Pfaffian graphs from r Pfaffian spanning subgraphs. Constructions of k-Pfaffian graphs known prior to this seem to be of a very different and essentially topological nature. We apply our Composition Theorem to produce a bipartite graph on 10 vertices that is 6-Pfaffian but not 4-Pfaffian. This is a counter-example to a conjecture of Norine (2009) [8], which states that the Pfaffian number of a graph is a power of four.

1. Introduction

Let G be a graph. Let $\{1, 2, \ldots, n\}$ be the set of vertices of G. For adjacent vertices u and v of G, we denote the edge joining u and v by uv or vu. Let D be an orientation of G. If D has an edge directed from u to v then we denote that directed edge by uv. Let $M := \{u_1v_1, u_2v_2, \ldots, u_kv_k\}$ be a perfect matching of D. Then the *sign of* M *in* D, denoted $\mathrm{sgn}(M, D)$, is the sign of the permutation

$$\pi_D(M) := \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & 2k-1 & 2k \\ u_1 & v_1 & u_2 & v_2 & \cdots & u_k & v_k \end{pmatrix}.$$

A change in the order of the enumeration of the edges of the perfect matching changes the number of inversions by an even number. Therefore, the sign of the permutation remains unchanged. We conclude that the sign of a perfect matching is well defined.

Let k be a positive integer, let $\boldsymbol{D} := (D_1, D_2, \ldots, D_k)$ be a k-tuple of orientations of G. We say that $\boldsymbol{D}$ is a *k-orientation* of G. For each perfect matching M of G, we associate the k-tuple

$$\mathbf{sgn}(M, \boldsymbol{D}) := (\mathrm{sgn}(M, D_1), \mathrm{sgn}(M, D_2), \ldots, \mathrm{sgn}(M, D_k)),$$

called the *signature vector of* M *relative to* $\boldsymbol{D}$. We denote by $\mathcal{M}(G)$, or simply $\mathcal{M}$, if G is understood, the set of perfect matchings of G. The *signature matrix of* $\mathcal{M}$ *relative to* $\boldsymbol{D}$ is the matrix

$$\mathbf{sgn}(\mathcal{M}, \boldsymbol{D}) := (\mathbf{sgn}(M, \boldsymbol{D}) : M \in \mathcal{M}).$$

If the system $\mathbf{sgn}(\mathcal{M}, \boldsymbol{D})\ \boldsymbol{x} = \mathbf{1}$ has a solution then we say that $\boldsymbol{D}$ is *Pfaffian* and, for any solution $\boldsymbol{\alpha}$ of that system, we say that $(\boldsymbol{D}, \boldsymbol{\alpha})$ is a *Pfaffian k-pair*. A Pfaffian k-pair $(\boldsymbol{D}, \boldsymbol{\alpha})$ of G is *normal* if $\boldsymbol{\alpha} > \mathbf{0}$.

We say that G is *k-Pfaffian* if it has a Pfaffian k-orientation. We note that relabelling the vertices of graph G either changes the signs of all perfect matchings relative to D or does not change the sign of any perfect matching of G relative to D.

* Corresponding author.
E-mail addresses: alberto.miranda@gmail.com (A.A.A. Miranda), lucchesi@gmail.com (C.L. Lucchesi).

0012-365X
doi:10.1016/j.disc.2010.10.014

Consequently, the property of G being k-Pfaffian does not depend on the particular enumeration of the vertices of G. We define the *Pfaffian number* of a graph G, denoted $\mathrm{pf}(G)$, to be the minimum k such that G is k-Pfaffian. A graph G is *Pfaffian* if $\mathrm{pf}(G) = 1$. Equivalently, a graph is Pfaffian if it admits an orientation in which all perfect matchings have the same sign.

Pfaffian graphs have applications in several areas, such as Physics, Chemistry and Economics (see for example the book by Lovász and Plummer [6, Chapter 8] and the paper by McCuaig [7]). Also, the problem of determining whether a directed graph has a cycle of even length is equivalent to that of determining whether a bipartite graph is Pfaffian. Robertson et al. [9] and, independently, McCuaig [7] have given a polynomial algorithm to solve this problem. In his book "Graph Theory As I Have Known It" [11], Tutte describes how he got the idea of using Pfaffians in order to obtain a formula to calculate the number of perfect matchings of a graph. The problem of determining the number of perfect matchings of a graph is equivalent to that of determining the permanent of a square matrix, which in turn was proved by Valiant to be NP-hard [13]. It is thus not surprising that Tutte did not succeed in establishing the formula, but nevertheless he was able to use Pfaffians in order to prove his celebrated theorem that characterizes graphs that have perfect matchings [12].

Kasteleyn [3] proved that every planar graph is Pfaffian (a proof may be found in the book by Lovász and Plummer [6, Theorem 8.3.4]). Galluccio and Loebl [1] and, independently, Tesler [10], proved a remarkable generalization of Kasteleyn's result:

Theorem 1. *If G is embeddable on an orientable surface of genus g then* $\mathrm{pf}(G) \leq 4^g$.

In fact, in 1967, Kasteleyn, in [2, page 99], stated a similar belief: "If the genus of the graph is g the number of Pfaffians required is 4^g". Later, in 2008, Norine [8] stated the following conjecture:

Conjecture 2. *The Pfaffian number of a graph is always a power of four.*

In partial support to his conjecture, Norine also proved the following result:

Theorem 3 (*Norine [8]*)**.** *Every* 3*-Pfaffian graph is Pfaffian, every* 5*-Pfaffian graph is* 4*-Pfaffian.*

The definition of Pfaffian number of a graph is based on properties depending on the perfect matchings of the graph. A similar definition may be made for even subgraphs of a graph. Recently, Loebl and Masbaum [5] proved that Conjecture 2 is true when we replace perfect matchings with even subgraphs in the definition of Pfaffian number.

2. Outline of the paper

2.1. r-decompositions

Let r be a positive integer, $G_1, G_2, \ldots, G_r$ Pfaffian spanning subgraphs of G. We say that $G_1, G_2, \ldots, G_r$ is an r-*decomposition* of a graph G if there are r sets $S_1, S_2, \ldots, S_r$ of edges of G such that

- $\{\mathcal{M}(G_i) : i = 1, 2, \ldots, r\}$ is a partition of $\mathcal{M}(G)$, and
- for each perfect matching M of G, $|M \cap S_i|$ is odd if and only if $M \in \mathcal{M}(G_i)$.

In Section 3 we prove the following basic result:

Theorem 11 (Composition)
If a graph has an r-decomposition then it is $2r$-Pfaffian.

2.2. Uniqueness of signature matrices

A Pfaffian 4-orientation of a 4-Pfaffian graph that is not Pfaffian is essentially unique. This fact is precisely specified by the following fundamental result, which is a restatement of Lemma 2.5 in Norine's paper [8].

Theorem 4 (*Uniqueness of Signature Matrices*)**.** *Let G be a non-Pfaffian graph, let $(\boldsymbol{D}, \boldsymbol{\alpha})$ be a normal Pfaffian* 4*-pair of G. Then,* $\boldsymbol{\alpha} = \mathbf{1/2}$.

2.3. Graphs G_{19} and G_{21}

Let us describe how a graph, which we call G_{19}, depicted in Fig. 1(a), is defined. Fig. 1(b) shows graph G_{21}. This graph is obtained from two disjoint copies, G_1 and G_2, of $K_{3,2}$, with sets of vertices X_1 and X_2, by joining every vertex of the majority part of G_1 to each vertex of the majority part of G_2. Those added edges span a $K_{3,3}$ and constitute a tight cut of G_{21}, which we denote by C_{21}. Graph G_{19} is obtained from G_{21} by removing two adjacent edges of C_{21}: the resulting tight cut is denoted by C_{19}.

In Section 4 we apply Theorem 11 to graph G_{21} and deduce that it is 6-Pfaffian. We show that a graph obtained from G_{21} by removing six edges from C_{21} so that the resulting tight cut spans a P_4 is Pfaffian. From this and Theorem 11 it follows that G_{21} is 6-Pfaffian, because it is possible to cover the edges of $K_{3,3}$ with three P_4's. It also follows that for every edge e in

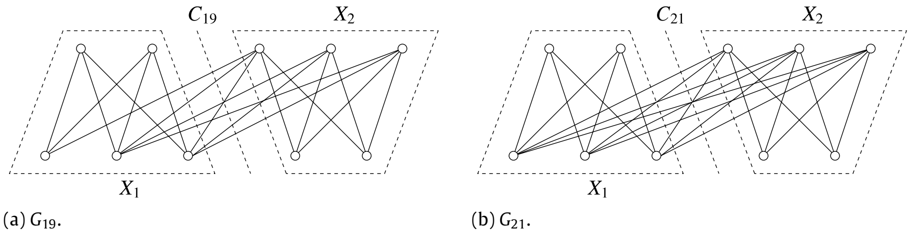


Fig. 1. Graphs G_{19} and G_{21}.

C_{19}, $G_{19} - e$ is 4-Pfaffian, because it is possible to cover $K_{3,3}$ minus any three edges with two P_4's. We also show that $G_{21} - e$ is 4-Pfaffian for any edge e not in C_{21}. This establishes the fact that $G_{19} - e$ is 4-Pfaffian, for every edge e. In sum, we prove that G_{19} is 6-Pfaffian, and if not 4-Pfaffian, then it is a minimal non-4-Pfaffian graph.

In Section 4 we also prove that G_{19} cannot possibly satisfy the property stated in Theorem 4. We deduce that G_{19} is 6-Pfaffian and minimal non-4-Pfaffian. Indeed, we believe that G_{19} is the smallest counter-example to Conjecture 2.

In Section 5 we present some conjectures on the Pfaffian number of graphs.

3. Composition of Pfaffian graphs

In this section we first establish a relation involving the Pfaffian numbers of the two C-contractions of a graph G and the Pfaffian number of G, where C is a tight cut of G. We then prove the Composition Theorem.

3.1. Edge cuts, similarity, normal pairs

Let G be a graph, X a set of vertices of G. We denote by $\partial(X)$ the (edge-)cut C consisting of those edges having one end in X, the other end in $\overline{X}$. The sets X and $\overline{X}$ are called *shores* of C. We say that two orientations D and D' of G are *similar* if the set of edges of G on which D and D' disagree is a cut of G. We say that two k-orientations $\boldsymbol{D}$ and $\boldsymbol{D}'$ of G are *similar* if there is a permutation f on $\{1, 2, \dots, k\}$ such that $\boldsymbol{D}_i$ and $\boldsymbol{D}_{f(i)}$ are similar, for $i = 1, 2, \dots, k$. For a directed graph D and a subset S of $E(D)$, let $D \otimes S$ denote the directed graph obtained from D by the reversal of the edges of S. The proof of the following result is straightforward:

Lemma 5. *Let D be a directed graph, M a perfect matching of D, and $C := \partial(X)$ a cut of D. Then,* $\mathrm{sgn}(M, D) = \mathrm{sgn}(M, D \otimes C)$ *if and only if $|X|$ is even.*

Corollary 6. *Let $\boldsymbol{D}$ and $\boldsymbol{D}'$ be two similar k-orientations of G. Then, $\boldsymbol{D}$ is Pfaffian if and only if $\boldsymbol{D}'$ is Pfaffian.*

Corollary 7. *Let $(\boldsymbol{D}, \boldsymbol{\alpha})$ be a Pfaffian k-pair of a graph G. Then, G has a Pfaffian k-pair $(\boldsymbol{D}', \boldsymbol{\alpha}')$ such that $\boldsymbol{D}$ and $\boldsymbol{D}'$ are similar and $\boldsymbol{\alpha}'_i = |\boldsymbol{\alpha}_i|$ for $i = 1, 2, \dots, k$.*

Corollary 8. *Every graph G has a normal Pfaffian* $\mathrm{pf}(G)$*-pair.*

3.2. Cut contractions and tight cuts

The graph obtained from X by contracting X to a single new vertex x and by removing any resulting loops is denoted by $G/X \to x$. The graphs $G/X \to x$ and $G/\overline{X} \to \overline{x}$ are called *C-contractions of G*. Assume further that G has a perfect matching. Cut C is *tight in G* if every perfect matching of G has precisely one edge in C. Little and Rendl [4] proved the following important result:

Theorem 9. *Let C be a tight cut of a graph G. Then, G is Pfaffian if and only if both C-contractions of G are Pfaffian.*

From Theorem 9 we deduce that if both C-contractions of G are Pfaffian and C is a tight cut then G is also Pfaffian. We need a generalization of this result for k-Pfaffian graphs. Theorem 9 does not extend naturally to k-Pfaffian graphs. Indeed, G_{21} is not 4-Pfaffian, yet both C_{21}-contractions of G_{21} are equal to $K_{3,3}$ up to multiple edges, whence 4-Pfaffian.

Theorem 10. *Let C be a tight cut of a graph G, let G' and G'' be the two C-contractions of G. Then,* $\mathrm{pf}(G) \le \mathrm{pf}(G') \cdot \mathrm{pf}(G'')$.

Proof. Let X denote the shore of C such that $G' = G/\overline{X} \to \overline{x}$ and $G'' := G/X \to x$. Let $(\boldsymbol{D}', \boldsymbol{\alpha}')$ be a Pfaffian $\mathrm{pf}(G')$-pair of G'. Adjust notation so that contraction vertex $\overline{x}$ has the highest label. Likewise, let $(\boldsymbol{D}'', \boldsymbol{\alpha}'')$ be a Pfaffian $\mathrm{pf}(G'')$-pair of G'', and adjust notation so that contraction vertex x has minimum label, equal to 1.

Let e_1 and e_2 denote any two multiple edges of G'. Denote by $\boldsymbol{D}' - e_2$ the pf(G')-orientation of $G' - e_2$ obtained by deleting the edge e_2 from each orientation of $\boldsymbol{D}'$. The pair $(\boldsymbol{D}' - e_2, \boldsymbol{\alpha}')$ is also a Pfaffian pf(G')-pair. Therefore, an extension of this pair to G', obtained by orienting e_2 in the same direction of e_1 in $\boldsymbol{D}_i' - e_2$, for $i = 1, 2, \ldots, \text{pf}(G')$, is also a Pfaffian pf($G'$)-pair. So, we may choose $(\boldsymbol{D}', \boldsymbol{\alpha}')$ such that every pair of multiple edges of G' has the same direction in each orientation of $\boldsymbol{D}'$. These observations imply that, for an orientation $\boldsymbol{D}_i'$ of $\boldsymbol{D}'$, the set S of edges of C directed away from contraction vertex $\bar{x}$ are part of a cut C' disjoint with $C - S$. We reverse the orientation of the edges of cut C' in $\boldsymbol{D}_i'$, thereby obtaining a similar orientation. We may thus assume that in $\boldsymbol{D}_i'$ all the edges of C enter contraction vertex $\bar{x}$. This conclusion holds for $i = 1, 2, \ldots, \text{pf}(G')$. Likewise, we may assume that each edge of cut C leaves x in $\boldsymbol{D}_j''$, for $j = 1, 2, \ldots, \text{pf}(G'')$. Define

$$\boldsymbol{D}_{ij} := \boldsymbol{D}_i' \cup \boldsymbol{D}_j'' \quad \text{and} \quad \boldsymbol{\alpha}_{ij} := \boldsymbol{\alpha}_i'\boldsymbol{\alpha}_j'', \quad \text{for } i = 1, 2, \ldots, \text{pf}(G') \text{ and } j = 1, 2, \ldots, \text{pf}(G'').$$

Label the vertices of G as follows: the vertices of X inherit their labels from G'; the vertices of $\overline{X}$ inherit their labels from G'', but are increased by $|X| - 1$. This clearly produces a labeling $1, 2, \ldots, |V(G)|$ of G. We assert that under this labeling, $(\boldsymbol{D}, \boldsymbol{\alpha})$ is a Pfaffian $\text{pf}(G') \cdot \text{pf}(G'')$-pair of G. For this, let M be a perfect matching of G. Then, $M' := M \cap E(G')$ is a perfect matching of G' and $M'' := M \cap E(G'')$ is a perfect matching of G''. The number of inversions of the permutation associated with M in $\boldsymbol{D}_{ij}$ is equal to the sum of the number of inversions of the permutations associated with M' in $\boldsymbol{D}_i'$ and M'' in $\boldsymbol{D}_j''$, for $i = 1, 2, \ldots, \text{pf}(G')$ and $j = 1, 2, \ldots, \text{pf}(G'')$. Thus, $\text{sgn}(M, \boldsymbol{D}_{ij}) = \text{sgn}(M, \boldsymbol{D}_i') \cdot \text{sgn}(M, \boldsymbol{D}_j'')$. Consequently,

$$\begin{aligned}\sum_{i,j} \boldsymbol{\alpha}_{ij} \,\text{sgn}(M, \boldsymbol{D}_{ij}) &= \sum_i \boldsymbol{\alpha}_i' \,\text{sgn}(M', \boldsymbol{D}_i') \sum_j \boldsymbol{\alpha}_j'' \,\text{sgn}(M'', \boldsymbol{D}_j'') \\ &= \sum_i \boldsymbol{\alpha}_i' \,\text{sgn}(M', \boldsymbol{D}_i') = 1.\end{aligned}$$

This conclusion holds for each perfect matching M of G. We deduce that, as asserted, $(\boldsymbol{D}, \boldsymbol{\alpha})$ is a Pfaffian $\text{pf}(G') \cdot \text{pf}(G'')$-pair of G. □

3.3. Composition of Pfaffian orientations

Theorem 11 (*Composition*). *If a graph has an r-decomposition then it is $2r$-Pfaffian.*

Proof. Let G be a graph, assume that G has an r-decomposition. Let $G_1, G_2, \ldots, G_r$ be Pfaffian spanning subgraphs of G, let $S_1, S_2, \ldots, S_r$ be sets of edges of G such that

- $\{\mathcal{M}(G_i) : i = 1, 2, \ldots, r\}$ is a partition of $\mathcal{M}(G)$, and
- for each perfect matching M of G, $|M \cap S_i|$ is odd if and only if $M \in \mathcal{M}(G_i)$.

Let us use the same labeling for each graph G_i and also for graph G. For $i = 1, 2, \ldots, r$, let D_i be a Pfaffian orientation of G_i. Adjust notation, by replacing D_i, if necessary, by $D_i \otimes \partial(v)$, for some vertex v of G, so that every perfect matching of G_i has sign equal to one in D_i. Let D_i' be an arbitrary extension of D_i to an orientation of G. Let $D_i'' := D_i' \otimes S_i$. Let

$$\boldsymbol{D} := (D_1', D_2', \ldots, D_r', D_1'', D_2'', \ldots, D_r''),$$
$$\boldsymbol{\alpha}_1 := \boldsymbol{\alpha}_2 := \cdots := \boldsymbol{\alpha}_r = 1/2 \quad \text{and} \quad \boldsymbol{\alpha}_{r+1} := \boldsymbol{\alpha}_{r+2} := \cdots := \boldsymbol{\alpha}_{2r} = -1/2.$$

We assert that $(\boldsymbol{D}, \boldsymbol{\alpha})$ is a Pfaffian $2r$-pair of G. For this, let M be a perfect matching of G. By hypothesis, M is a perfect matching of precisely one of the graphs G_i, say G_k. By hypothesis, $|M \cap S_i|$ is odd if and only if $i = k$. For $i \neq k$, as $|M \cap S_i|$ is even, it follows that M has equal signs in D_i' and in D_i''. Consequently, $\text{sgn}(M, \boldsymbol{D}) \cdot \boldsymbol{\alpha} = 1/2[\text{sgn}(M, D_k') - \text{sgn}(M, D_k'')]$. As $|M \cap S_k|$ is odd, we have that

$$\text{sgn}(M, D_k') = \text{sgn}(M, D_k) = 1 \quad \text{and} \quad \text{sgn}(M, D_k'') = -\text{sgn}(M, D_k) = -1.$$

Consequently, $\text{sgn}(M, \boldsymbol{D}) \cdot \boldsymbol{\alpha} = 1$. This conclusion holds for each perfect matching M of G. As asserted, G is $2r$-Pfaffian. □

Corollary 12. *Let C be a tight cut of a graph G. Let $\{C_1, C_2, \ldots, C_r\}$ be a partition of C. Assume that $G_i := G-(C-C_i)$ is Pfaffian, for $i = 1, 2, \ldots, r$. Then, G is $2r$-Pfaffian.*

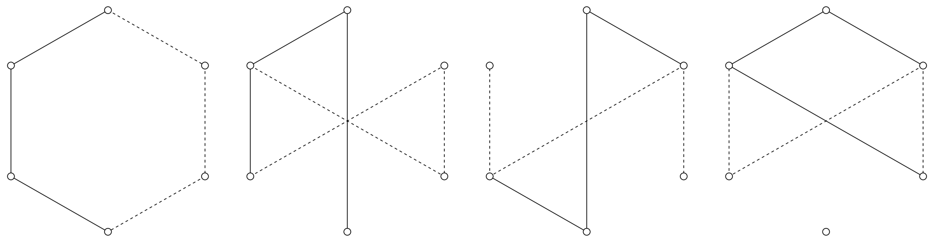
4. Applications

4.1. Graph G_{21} is 6-Pfaffian and graph $G_{19} - e$ is 4-Pfaffian

In this section we prove that graph G_{21} is 6-Pfaffian. Consequently, G_{19} is also 6-Pfaffian. We also prove that $G_{19} - e$ is 4-Pfaffian, for every edge e. We begin by deriving a straightforward consequence of Corollary 12.

Theorem 13. *Let R be a (possibly empty) subset of tight cut C_{21} of G_{21}. If $C_{21} - R$ may be covered by r edge-disjoint P_4's then $G_{21} - R$ is $2r$-Pfaffian.*

Proof. Assume that $C_{21} - R$ is covered by r P_4's. Let $C_1, \ldots, C_r$ denote the set of edges of the r P_4's. By Corollary 12, it suffices to show that $G_i := G_{21} - (C - C_i)$ is Pfaffian, for $i = 1, \ldots, r$. For this, note that the C_i-contractions of G_i are equal to $K_{3,3} - e$,



(a) R is a matching. (b) Exactly two edges of R are adjacent. (c) R spans P_4. (d) R spans a star.

Fig. 2. Decomposition of $K_{3,3} - R$ in two P_4's, where R is a set of three edges.

up to multiple edges. As $K_{3,3} - e$ is planar, it is Pfaffian. Therefore, both C_i-contractions of G_i are Pfaffian. Moreover, C_i is tight in G_i. We deduce that G_i is Pfaffian. The assertion holds. □

Theorem 14. *Graph G_{21} is 6-Pfaffian and graph $G_{19} - e$ is 4-Pfaffian.*

Proof. Note that $G_{21}[C_{21}]$ is $K_{3,3}$ (see Fig. 1(b)). Let R be any set of three edges of $K_{3,3}$. As indicated in Fig. 2, $K_{3,3} - R$ is the union of two P_4's.

In particular, if R is the set of edges of a P_4 of $K_{3,3}$, we deduce that $K_{3,3}$ is the union of three P_4's. By Theorem 13, graph G_{21} is 6-Pfaffian. For any edge e of C_{19}, the cut $C_{19} - e$ spans a $K_{3,3}$ minus three edges. In this case, $G_{19} - e$ is 4-Pfaffian, by Theorem 13. Finally, if e is an edge of G_{21} that does not lie in C_{21}, then, up to multiple edges, one of the C_{21}-contractions of $G_{21} - e$ is $K_{3,3} - e$, the other C-contraction is $K_{3,3}$. As $K_{3,3}$ is 4-Pfaffian and $K_{3,3} - e$ is Pfaffian, it follows that $G_{21} - e$ is 4-Pfaffian. We deduce that $G_{19} - e$ is also 4-Pfaffian. This conclusion holds for each edge e of G_{19}. The assertion holds. □

4.2. Graph G_{19} is not 4-Pfaffian

For directed graph D, a cycle Q of even length of D is *evenly oriented* if the number of forward edges of Q is even, *oddly oriented* otherwise. The following result appears in the book by Lovász and Plummer [6, Lemma 8.3.1]:

Lemma 15. *Let M_1 and M_2 be two perfect matchings of a directed graph D, and let k denote the number of evenly oriented M_1, M_2-alternating cycles. Then, $\mathrm{sgn}(M_1, D) \cdot \mathrm{sgn}(M_2, D) = (-1)^k$.*

Theorem 16. *Graph G_{19} is not 4-Pfaffian.*

Proof. Refers to Fig. 1. Let us first prove that G_{19} is not Pfaffian. For this, note that C_{19} is tight and each C_{19}-contraction of G_{19} is, up to multiple edges, equal to $K_{3,3}$, in turn non-Pfaffian. Therefore, by Theorem 9, G_{19} is not Pfaffian.

Assume, to the contrary, that G_{19} is 4-Pfaffian. Let $(\boldsymbol{D}, \boldsymbol{\alpha})$ be a normal Pfaffian 4-pair of G. Let $\boldsymbol{S}$ be the signature matrix $\mathbf{sgn}(\mathcal{M}(G), \boldsymbol{D})$. By the theorem on the Uniqueness of Signature Matrices, we have that $\boldsymbol{\alpha} = 1/2$, whence every row of $\boldsymbol{S}$ contains precisely one entry equal to -1. Moreover, every column of $\boldsymbol{S}$ contains one entry equal to -1. The set $\mathcal{M}$ of the perfect matchings of G is thus partitioned in four non-null classes, $\mathcal{M}_i$, $i = 1, 2, 3, 4$, such that $\mathcal{M}_i$ is the set of those perfect matchings of G that have sign -1 in $\boldsymbol{D}_i$ (and sign 1 in all the other three orientations in $\boldsymbol{D} - \boldsymbol{D}_i$). Let us now derive some properties of $\boldsymbol{D}$. Recall first that $G_{19} - C_{19}$ is the union of two disjoint $K_{3,2}$'s, $G_i := G_{19}[X_i]$, $i = 1, 2$ (see Fig. 1). The following auxiliary result is easily proved:

Lemma 17. *In every orientation of $K_{3,2}$, the number of evenly oriented cycles is odd.*

Lemma 18. *Let Q be a quadrilateral in $G_{19} - C_{19}$ that is evenly oriented in $\boldsymbol{D}_i$. Then, Q is evenly oriented in precisely one more orientation $\boldsymbol{D}_j$ of G, $j \neq i$. Moreover, every perfect matching of G that contains two edges in Q lies in $\mathcal{M}_i \cup \mathcal{M}_j$.*

Proof. Let M be a perfect matching of G that contains two edges in Q. Let $N := M \triangle Q$. The signs of M and N in $\boldsymbol{D}_i$ are distinct. Therefore, one of M and N has sign -1 in $\boldsymbol{D}_i$, the other has sign 1 in $\boldsymbol{D}_i$. Consequently, there exists an integer j distinct from i such that one of M and N has sign -1 in $\boldsymbol{D}_i$ and sign 1 in $\boldsymbol{D}_j$, the other has sign 1 in $\boldsymbol{D}_i$ and sign -1 in $\boldsymbol{D}_j$. We deduce that Q is evenly oriented in $\boldsymbol{D}_j$ as well. For any orientation $\boldsymbol{D}_k$ in $\boldsymbol{D} - \boldsymbol{D}_i - \boldsymbol{D}_j$, the signs of M and N in $\boldsymbol{D}_k$ are both equal to 1. Therefore, Q is oddly oriented in $\boldsymbol{D}_k$. We deduce that Q is oddly oriented in the two orientations of $\boldsymbol{D} - \boldsymbol{D}_i - \boldsymbol{D}_j$ and evenly oriented in $\boldsymbol{D}_i$ and in $\boldsymbol{D}_j$. Finally, we have already seen that M lies in $\mathcal{M}_i \cup \mathcal{M}_j$. This conclusion holds for each perfect matching M of G that contains two edges in Q. □

Corollary 19. *For each shore X_i of C, at most one of the three cycles in $G_{19}[X_i]$ is oddly oriented in every orientation in $\boldsymbol{D}$.*

Proof. Let r denote the number of cycles of $G_{19}[X_i]$ that are evenly oriented in some orientation in $\boldsymbol{D}$. By Lemma 18, each such cycle is evenly oriented in precisely two orientations. A simple counting argument then shows that the number of pairs

$(Q, \boldsymbol{D}_j)$ such that Q is a cycle of $G_{19}[X_i]$ that is evenly oriented in $\boldsymbol{D}_j$ is equal to $2r$. Every orientation contains at least one evenly oriented cycle in $G_{19}[X_i]$. We deduce that $2r \geq 4$, whence $r \geq 2$. As asserted, at most one of the three quadrilaterals of $G_{19}[X_i]$ is oddly oriented in every orientation in $\boldsymbol{D}$. $\square$

Let x_1 and x_2 denote the two *universal* vertices of X_1, that is, the two vertices of degree five in G_{19} that lie in X_1. For $i = 1, 2$, subgraph $G_{19}[X_1]$ of G_{19} has two cycles, Q_1 and Q_2, such that Q_i contains x_i but does not contain both x_1 and x_2. By the corollary, at least one of Q_1 and Q_2 is evenly oriented in some orientation in $\boldsymbol{D}$. Adjust notation so that x is a universal vertex of X_1, Q is a cycle of $G_{19}[X_1] - x$ that is evenly oriented in $\boldsymbol{D}_1$. Adjust notation so that Q is evenly oriented in $\boldsymbol{D}_2$ as well. Then, Q is oddly oriented in $\boldsymbol{D}_3$ and in $\boldsymbol{D}_4$.

Let Q' denote a cycle in $G_{19}[X_2]$ that is evenly oriented in $\boldsymbol{D}_3$. Then, Q' is also evenly oriented in $\boldsymbol{D}_j$, for some j in $\{1, 2, 4\}$, but oddly oriented in the two orientations in $\boldsymbol{D} - \boldsymbol{D}_3 - \boldsymbol{D}_j$.

Let e be the edge of $G_{19} - V(Q) - V(Q')$. That edge exists, because the vertex x in $X_1 - V(Q)$ is universal. Let M be a perfect matching of G_{19} that contains edge e. Then, M contains two edges in Q and two edges in Q'. Let $N := M \Delta E(Q)$. By the lemma, $\{M, N\} \subset \mathcal{M}_1 \cup \mathcal{M}_2$. Again, by the lemma, $\{M, N\} \subset \mathcal{M}_3 \cup \mathcal{M}_j$, which implies that M and N lie both in $\mathcal{M}_j$. But M and N have distinct signs in $\boldsymbol{D}_1$, whence cannot both lie in the same class $\mathcal{M}_j$. We have deduced a contradiction from the hypothesis that G_{19} is 4-Pfaffian. $\square$

5. Final remarks

Norine [8] has shown that for every graph G, if $\mathrm{pf}(G) \leq 5$ then $\mathrm{pf}(G) \in \{1, 4\}$. We have shown that G_{19} is a graph that is 6-Pfaffian but not 4-Pfaffian. This is a counter-example to Conjecture 2. We believe it to be a minimum counter-example.

We have written Python programs to check the Pfaffian number of relatively small graphs, by brute force, which demanded enormous computing resources. We have never found graphs with odd Pfaffian numbers. On the basis of that empirical evidence, plus the results proved by Norine, and our Composition Theorem, we are led to believe that the Pfaffian number of every graph is even and we propose the following conjecture:

Conjecture 20. *For every graph G, if* $\mathrm{pf}(G) > 1$ *then* $\mathrm{pf}(G)$ *is even. Moreover, for every even integer* $k \geq 4$ *there exists a graph G whose Pfaffian number is k.*

Acknowledgements

We would like to thank the referees for important suggestions that shortened the paper and also for mentioning relevant citations that were missing in the original draft.

The first author's research was supported by FAPESP — Fundação de Amparo à Pesquisa do Estado de São Paulo. The second author's research was supported in part by CNPq — Conselho Nacional de Desenvolvimento Científico e Tecnológico.

References

[1] A. Galluccio, M. Loebl, On the theory of Pfaffian orientations. I. Perfect matchings and permanents, Electron. J. Combin. 6 (1999).
[2] F. Harary (Ed.), Graph Theory and Theoretical Physics, Academic Press Inc., 1967.
[3] P.W. Kasteleyn, Dimer statistics and phase transitions, J. Math. Phys. 4 (1963) 287–293.
[4] C.H.C. Little, F. Rendl, Operations preserving the Pfaffian property of a graph, J. Aust. Math. Soc. Ser. A 50 (1991) 248–275.
[5] M. Loebl, G. Masbaum, On the optimality of the arf invariant formula for graph polynomials, Adv. Math. 226 (2011) 332–349.
[6] L. Lovász, M.D. Plummer, Matching Theory, in: Annals of Discrete Mathematics, vol. 29, Elsevier Science, 1986.
[7] W. McCuaig, Pólya's permanent problem, Electron. J. Combin. 11 (2004).
[8] S. Norine, Drawing 4-Pfaffian graphs on the torus, Combinatorica 29 (2009) 109–119.
[9] N. Robertson, P.D. Seymour, R. Thomas, Permanents, Pfaffian orientations and even directed circuits, Ann. of Math. (2) 150 (1999) 929–975.
[10] G. Tesler, Matchings in graphs on non-orientable surfaces, J. Combin. Theory Ser. B 78 (2000) 198–231.
[11] W.T. Tutte, Graph Theory as I Have Known It, in: Oxford Lecture Series in Mathematics and its Applications, vol. 11, Clarendon Press, Oxford, 1998.
[12] W.T. Tutte, The factorization of linear graphs, J. Lond. Math. Soc. 22 (1947) 107–111.
[13] L. Valiant, The complexity of computing the permanent, Theoret. Comput. Sci. 8 (1979) 189–201.